Lecture 2b : *FD Methods for Elliptic Equations: Error Analysis*

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References and Acknowledgements

The following materials were used in the preparation of this lecture:

1. Tannehill, Anderson and Pletcher, *Computational fluid Mechanics and Heat Transfer*.
2. 16.920, lecture 2,3,4 Notes

The author of these slides wishes to thank these sources for making the current lecture.

Finite Differences and Convergence

- Consider 1D and 2D eliptic finite difference methods:
  1. Do these methods converge to a single answer?
  2. Is this convergence guaranteed?
  3. Can we say anything about the error?
- Consider the 1D Laplace equation example, we can analytically compute the exact answer.
- Let's write the analytical solution below:

Finite Differences and Convergence

- We will compute the difference between the analytical solution and the numerically computed solution
  - At specific points in the domain (individual points, because this is where the solution is being determined).
  - Number of nodes $= (n+1)$
  - Let's examine the ∞-norm of the error

Finite Differences and Convergence

- The parameters for the error are:
  - $error = \|u - \hat{u}\|_\infty \simeq C\Delta x^\alpha$?
  - $\hat{u}$ is the numerical approximation to the solution

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General Convergence Analysis – Elliptic Equations

- There are two *fundamental* conditions for convergence:
  1. **Consistency** : (elliptical problem) A numerical approximation is consistent if, for all smooth solutions, the numerical approximation $\hat{u}$ tends toward the theoretical answer u.
  2. **Stability** : (elliptical problem) Stability implies a numerical approximation that does not amplify error or perturbations in the RHS.
- So, **convergence** = **stability** + **consistency**
- Let's look at each component in a bit more depth.

Consistency

- Consider a PDE that is written as:

$$Lu = f \tag{1}$$

- Consistency examines the difference between the numerical approximation and the actual solution:

$$\left(\hat{L}u - \hat{f}\right)_j - (Lu - f)_j \;\; = \;\; Order(\Delta x^p) \rightarrow 0 \tag{2}$$

- for all $j = 1, 2, 3, ..., n$ as $\Delta x \rightarrow 0$.
  - $\hat{\cdot}$ indicates the numerical approximation
  - p here is the order of accuracy
  - u is an arbitrary exact solution to the system

Consistency

- The equation can be simplified to give some insight into the truncation error, τ (recall we truncated the Taylor Series):

$$\underbrace{\left(\hat{L}u-\hat{f}\right)_j}_{DiscreteOperator}-\underbrace{\left(Lu-f\right)_j}_{=0}=\tau_j \tag{3}$$

- The goal is to have $\tau \rightarrow 0$ as $\Delta x \rightarrow 0$:

$$\left(\hat{L}u\right) = \tau+\hat{f} \tag{4}$$

$$but,\ \ \hat{f} = \left(\hat{L}\hat{u}\right),\ \ hence, \tag{5}$$

$$\left(\hat{L}u\right) = \tau+\hat{L}\hat{u} \tag{6}$$

$$\left(\hat{L}(\underbrace{u-\hat{u}}_{e})\right) = \left(\hat{L}e\right)=\tau \tag{7}$$

Consistency

- There is a direct link between the T.S. truncation error τ and the solution error e:

$$\left(\hat{L}e\right) = \tau \tag{8}$$

- Taylor-Series $\rightarrow$ truncation error "rate" (eg: $O(\Delta x^2)$).
- GOAL: Find $e = A^{-1}\tau$ to see the magnitude of error in discretization.
  - ie. How well A^{-1} is behaved will dictate error magnitude.

Stability

- **Stability**: If the solution perturbations do not grow as a function of Δx then the numerical scheme is stable.
- This can be written mathematically as:

$$\begin{aligned} \hat{L}u &= \hat{f} \\ u &= \hat{L}^{-1}\hat{f} \\ \|L^{-1}\|_\infty &\leq C \end{aligned}$$

- C: Is a constant that is *independent* of Δx
- Stability $\rightarrow$ matrix is not magnifying the RHS as we change Δx
- *It turns out* that (see MIT notes, pg 17 Lec 2&3), $\|L^{-1}\|_\infty$ is simply the max row sum of L^{-1}.

Stability: Example

- Let's look at A^{-1} for the string problem. What is the maximum row sum for different numbers of nodes?

Convergence – Page 18 MIT notes

- This is a neat result, now that we know more about stability and consistency:

$$\begin{aligned} e &= \hat{L}^{-1}\tau && (10) \\ \|e\|_\infty &= \|\hat{L}^{-1}\tau\|_\infty && (11) \\ &= \|\hat{L}^{-1}\|_\infty \|\tau\|_\infty && (12) \\ &\leq \underbrace{C\cdot}_{Stability} \underbrace{\Delta x^p}_{Consistency} && (13) \end{aligned}$$

- You can **also** use *eigenvalue analysis* to examine convergence of elliptic systems. This is not covered in this course.